# Canonical Ensemble of Noninteracting Bosonic Atoms and Interference of Bose-Einstein Condensates

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## 1 Introduction

I met Professor Shuichi Tasaki for the first time at a summer school held at Trento, Italy, in 1999, when I was a doctor-course student at Waseda University. Then, in the next spring, I found him with surprise at Waseda University as a new faculty member of Department of Applied Physics. He acted as a reviewer of my doctoral thesis and was my advisor for my current position at Waseda University. I am very happy that I could work with Prof. Tasaki and could publish four papers with him. He always tried to be available for discussion and gave me a lot of suggestions. I got many hints from the collaborations, from his lectures, and from daily conversations with him. Many of my recent works do not exist without the fruitful interactions with Prof. Tasaki. While he had to stay in hospital, I could just push forward with my researches, to be ready to report good results to him as soon as he is back in his office. I regret that I was unable to do it before he passed away, but I would like to take this opportunity to report one of the results I wanted to show him.

Prof. Tasaki had a quite broad spectrum of interests and knowledge, mainly in the fields of statistical mechanics, condensed-matter physics, and quantum physics, which should be clear from the list of the authors of the present volume. He was not only an excellent physicist but also a brilliant mathematician. He was a rare scientist who could talk with both physicists and mathematicians. He proved purely mathematical theorems rigorously, and he liked exact solutions, but he did not hesitate to make drastic approximations and got through "dirty" calculations to explain experimental data. Mathematics spoken by Prof. Tasaki was very clear to physicists. Prof. Tasaki was in particular interested in the $C^*$-algebraic approach to the statistical and quantum mechanics. Several years ago, he gave a series of lectures on the $C^*$-algebra, which was not an authorized course of the university but was delivered mainly for the students of his group. The note I took during this lecture course always helps me when I am lost in the mathematics of the $C^*$-algebra. In addition, the note held a clue to my recent work with Mauro Iazzi on the interference of independent Bose-Einstein condensates [1].

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## 2 Canonical vs Grand Canonical Ensembles of Free Bosons

The relevant topic is the *canonical* vs *grand canonical* ensembles of free bosons [2]. Consider an ideal gas of bosons in a box of size $L$ with the periodic boundary condition at a finite temperature $T$. If one describes it as a grand canonical ensemble, the expectation value of an observable $\hat{\mathcal{O}}$ is given by

$$\langle\hat{\mathcal{O}}\rangle_G = \frac{\mathrm{Tr}\{\hat{\mathcal{O}}e^{-\beta(\hat{H}-\mu\hat{N})}\}}{\mathrm{Tr}\{e^{-\beta(\hat{H}-\mu\hat{N})}\}} \tag{2.1}$$

with

$$\hat{H} = \sum_{\boldsymbol{k}} \varepsilon_k \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}}, \qquad \varepsilon_k = \frac{\hbar^2 k^2}{2m}, \qquad \hat{N} = \sum_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}}, \tag{2.2}$$

where $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}_{\boldsymbol{k}}^\dagger$ are the annihilation and the creation operators for a boson of momentum $\boldsymbol{k}$, satisfying the canonical commutation relations $[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger] = \delta_{\boldsymbol{k}\boldsymbol{k}'}$, etc., and $\beta = 1/k_B T$ is the inverse temperature. $\mu$ is the chemical potential to be fixed by the condition that the average number of bosons in the gas is $N$, i.e.,

$$\langle\hat{N}\rangle_G = \sum_{\boldsymbol{k}} \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1} = N. \tag{2.3}$$

We take the thermodynamical limit $L, N \to \infty$ with the density $\rho = N/L^3$ kept finite. The system exhibits phase transition and a macroscopic number $N_0$ of bosons start to occupy the ground state as [3]

$$\rho_0 = \frac{N_0}{L^3} = \begin{cases} \rho\left[1-\left(\dfrac{T}{T_c}\right)^{3/2}\right] & (T \le T_c,\ \mu = 0), \\ 0 & (T > T_c,\ \mu < 0), \end{cases} \tag{2.4}$$

below the critical temperature $T_c = (2\pi\hbar^2/mk_B)[\rho/\zeta(3/2)]^{2/3}$, where $\zeta(z) = \sum_{n=1}^\infty n^{-z}$ ($\mathrm{Re}\,z > 1$) is the Riemann zeta function.

This ensemble is fully characterized by the generating functional for normally-ordered products, namely, the expectation value of

$$\hat{W}[J, J^*] = e^{i\sum_{\boldsymbol{k}} J_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}^\dagger} e^{i\sum_{\boldsymbol{k}} J_{\boldsymbol{k}}^*\hat{a}_{\boldsymbol{k}}}, \tag{2.5}$$

which reads

$$W_G[J, J^*] = \langle\hat{W}[J, J^*]\rangle_G = \exp\left(-\sum_{\boldsymbol{k}} \frac{|J_{\boldsymbol{k}}|^2}{e^{\beta(\varepsilon_k-\mu)}-1}\right). \tag{2.6}$$

In the thermodynamical limit, it is reduced to

$$W_G[J, J^*] \to \mathcal{W}_G[\mathcal{J}, \mathcal{J}^*] = \Xi_G(\mathcal{J}_0, \mathcal{J}_0^*) \exp\left(-\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{|\mathcal{J}_{\boldsymbol{k}}|^2}{e^{\beta(\varepsilon_k-\mu)}-1}\right), \tag{2.7}$$

with a factor relevant to the condensate,

$$\Xi_G(\mathcal{J}_0,\mathcal{J}_0^*)=\begin{cases}e^{-\rho_0|\mathcal{J}_0|^2} & (T\le T_c,\ \mu=0),\\ 1 & (T>T_c,\ \mu<0),\end{cases}\tag{2.8}$$

where $J_{\boldsymbol{k}}$ has been scaled as $J_{\boldsymbol{k}}=\mathcal{J}_{\boldsymbol{k}}/L^{3/2}$.

If, on the other hand, one describes the gas as a canonical ensemble with the number of atoms fixed at $N$, the expectation value of an observable $\hat{\mathcal{O}}$ is given by

$$\langle\hat{\mathcal{O}}\rangle_N=\frac{\mathrm{Tr}\{\hat{\mathcal{O}}\hat{P}_N e^{-\beta\hat{H}}\}}{\mathrm{Tr}\{\hat{P}_N e^{-\beta\hat{H}}\}},\tag{2.9}$$

instead of (2.1) for the grand canonical ensemble, where $\hat{P}_N$ is the projection operator that selects the $N$-particle sector, satisfying $\hat{P}_N\hat{P}_{N'}=\delta_{NN'}\hat{P}_N$ and $\sum_N\hat{P}_N=1$. It is not quite easy to directly compute this expectation value (2.9) for the canonical ensemble. It is however attainable via inverting the relationship between the canonical and the grand canonical ensembles,

$$\langle\hat{\mathcal{O}}\rangle_G=\sum_{N'}\langle\hat{P}_{N'}\rangle_G\langle\hat{\mathcal{O}}\rangle_{N'}.\tag{2.10}$$

In the thermodynamical limit, this relation (2.10) for the characteristic functionals $W_G[J,J^*]$ and $W_N[J,J^*]=\langle\hat{W}[J,J^*]\rangle_N$ is reduced to

$$\mathcal{W}_G[\mathcal{J},\mathcal{J}^*]=\int_0^\infty d\rho'\,\mathcal{K}(\rho')\mathcal{W}_{\rho'}[\mathcal{J},\mathcal{J}^*],\tag{2.11}$$

where

$$W_N[J,J^*]\to\mathcal{W}_\rho[\mathcal{J},\mathcal{J}^*],\qquad L^3\langle\hat{P}_{N'}\rangle_G\to\mathcal{K}(\rho'),\qquad \rho'=N'/L^3.\tag{2.12}$$

By noting

$$\begin{aligned}L^3\langle\hat{P}_{N'}\rangle_G&=L^3\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\langle e^{i\theta(N'-\hat{N})}\rangle_G\\&=\int_{-\pi L^3}^{\pi L^3}\frac{d\xi}{2\pi}e^{i\xi\rho'}\langle e^{-i\xi\hat{N}/L^3}\rangle_G\to\begin{cases}\displaystyle\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\frac{e^{i\xi(\rho'-\bar{\rho})}}{1+i\xi\rho_0} & (T\le T_c,\ \mu=0),\\ \displaystyle\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}e^{i\xi(\rho'-\rho)} & (T>T_c,\ \mu<0),\end{cases}\end{aligned}\tag{2.13}$$

the kernel $\mathcal{K}(\rho')$ is estimated to be

$$\mathcal{K}(\rho')=\begin{cases}\theta(\rho'-\bar{\rho})\dfrac{1}{\rho_0}e^{-(\rho'-\bar{\rho})/\rho_0} & (T\le T_c,\ \mu=0),\\ \delta(\rho'-\rho) & (T>T_c,\ \mu<0),\end{cases}\tag{2.14}$$

where $\bar{\rho}=\rho-\rho_0$. For $T\le T_c$, the above relation (2.11) yields

$$\int_0^\infty d\rho'\,e^{-\rho'/\rho_0}\mathcal{W}_{\rho'+\bar{\rho}}[\mathcal{J},\mathcal{J}^*]=\rho_0 e^{-\rho_0|\mathcal{J}_0|^2}\exp\biggl(-\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{|\mathcal{J}_{\boldsymbol{k}}|^2}{e^{\beta\varepsilon_k}-1}\biggr),\tag{2.15}$$

which is essentially the Laplace transformation from $\rho'$ to $1/\rho_0$. Its inversion leads to [2]

$$\mathcal{W}_\rho[\mathcal{J},\mathcal{J}^*] = \Xi_\rho(\mathcal{J}_0,\mathcal{J}_0^*)\exp\biggl(-\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{|\mathcal{J}_{\boldsymbol{k}}|^2}{e^{\beta(\varepsilon_k-\mu)}-1}\biggr), \tag{2.16}$$

$$\Xi_\rho(\mathcal{J}_0,\mathcal{J}_0) = \begin{cases} J_0(2\sqrt{\rho_0}|\mathcal{J}_0|) & (T\le T_c,\ \mu=0),\\ 1 & (T>T_c,\ \mu<0),\end{cases} \tag{2.17}$$

where

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \int_{\sigma-i\infty}^{\sigma+i\infty}\frac{ds}{2\pi i}s^{-\nu-1}e^{s-z^2/4s}\quad(\operatorname{Re}\nu>0,\ \sigma>0) \tag{2.18}$$

is a Bessel function.

Above the critical temperature $T>T_c$, both characteristic functionals (2.7) and (2.16) for the grand canonical and the canonical ensembles coincide, while they differ below $T\le T_c$. In addition, it is instructive to decompose the characteristic functional (2.16) for the canonical ensemble below the critical temperature $T\le T_c$ as

$$\mathcal{W}_\rho[\mathcal{J},\mathcal{J}^*] = \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\mathcal{W}_\theta[\mathcal{J},\mathcal{J}^*],\quad \mathcal{W}_\theta[\mathcal{J},\mathcal{J}^*] = e^{2i\operatorname{Re}(\sqrt{\rho_0}e^{i\theta}\mathcal{J}_0^*)}\exp\biggl(-\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{|\mathcal{J}_{\boldsymbol{k}}|^2}{e^{\beta(\varepsilon_k-\mu)}-1}\biggr), \tag{2.19}$$

by noting the formula for the Bessel function

$$J_n(x) = \frac{1}{\pi i^n}\int_0^\pi d\theta\, e^{ix\cos\theta}\cos n\theta\quad(x\in\mathbb{R},\ n=0,1,2,\ldots). \tag{2.20}$$

In particular, the characteristic functional $\mathcal{W}_\theta[\mathcal{J},\mathcal{J}^*]$ generates an expectation value of $\hat{a}_0$ as

$$\langle\hat{a}_0\rangle_\theta = \sqrt{N_0}e^{i\theta}, \tag{2.21}$$

which is the so-called *wave function of the condensate*, endowed with a definite phase $\theta$ [4]. The canonical ensemble $\mathcal{W}_\rho[\mathcal{J},\mathcal{J}^*]$ is a mixture of such condensed states with different phases.

This knowledge I acquired from Prof. Tasaki's lecture helped me to study the interference of independent Bose-Einstein condensates.

# 3 Interference of Two Independent Bose-Einstein Condensates

When two independently prepared Bose-Einstein condensates are released from traps and overlap, interference fringes are observed [5]. The simplest description of this phenomenon is based on the *spontaneous symmetry breaking* [4]. The U(1) symmetry of the system is spontaneously broken upon condensation and the two gases individually acquire definite phases. As a result, the relative phase between the gases becomes well defined, which enables them to exhibit interference. The symmetry breaking, however, would be valid only approximately, since the actual gases in typical interference experiments consist of finite numbers of atoms. Javanainen and Yoo showed in [6] that, even if the number of atoms in each gas is fixed, described by a number state, and the phase of each gas is accordingly uncertain, an interference pattern is observed in each *snapshot* photo of the overlapping gases. The density profiles would differ from snapshot to snapshot, and the appearance of an interference pattern is not definitely certain. According to the numerical simulation by Javanainen and Yoo in [6], however, sinusoidal patterns are very *typical* among all possible snapshot profiles and interference is *almost certainly* observed in *every* snapshot. One of the interference patterns, in other words, one definite relative phase, is selected by taking a photo, i.e., by measurement, and such interference revealed in a snapshot is called *measurement-induced interference* [7, 8].

I have found the subject interesting, since it requires studying *snapshots* in quantum mechanics, and got the following idea to tackle this issue.

### 3.1 Average Fringe Spectrum and Its Fluctuation

Our idea is similar to the one employed in [9–11]. Suppose that there are $N$ bosonic atoms and one takes a photo of the cloud: the positions of the $M$ atoms (generally smaller than $N$) are recorded at once in the snapshot. The probability of finding the $M$ atoms (among $N$) at positions $\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M\}$ at an instant $t$ is given by the $M$-particle distribution function

$$P_t^{(M)}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) = \frac{(N-M)!}{N!} \langle \hat{\psi}^\dagger(\boldsymbol{r}_1) \cdots \hat{\psi}^\dagger(\boldsymbol{r}_M) \hat{\psi}(\boldsymbol{r}_M) \cdots \hat{\psi}(\boldsymbol{r}_1) \rangle_t, \tag{3.22}$$

where $\hat{\psi}(\boldsymbol{r})$ is the field operator of the bosonic atom, satisfying the canonical commutation relations $[\hat{\psi}(\boldsymbol{r}), \hat{\psi}^\dagger(\boldsymbol{r}')] = \delta^3(\boldsymbol{r} - \boldsymbol{r}')$, etc., and $\langle \cdots \rangle_t$ denotes the expectation value estimated in the state of the cloud at time $t$. This probability is normalized to unity as

$$\int d^3\boldsymbol{r}_1 \cdots d^3\boldsymbol{r}_M \, P_t^{(M)}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) = 1, \tag{3.23}$$

and satisfies the recursive relation

$$\int d^3\boldsymbol{r}_\ell \, P_t^{(M)}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_\ell, \ldots, \boldsymbol{r}_M) = P_t^{(M-1)}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{\ell-1}, \boldsymbol{r}_{\ell+1}, \ldots, \boldsymbol{r}_M). \tag{3.24}$$

In the following, we assume that all of the $N$ atoms are recorded in the snapshot photo.

Given a single configuration of the $N$ atoms $\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N\}$ in a photo, the snapshot density profile of the cloud is given by

$$\rho(\boldsymbol{r}) = \frac{1}{N} \sum_{i=1}^{N} \delta(\boldsymbol{r} - \boldsymbol{r}_i). \tag{3.25}$$

Notice that the positions of the $N$ atoms, $\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N\}$, differ from run to run, and the density profile $\rho(\boldsymbol{r})$ changes from snapshot to snapshot. The average profile over all possible configurations of the $N$ atoms (over all snapshots) is given by

$$\overline{\rho(\boldsymbol{r})} = \int d^3\boldsymbol{r}_1 \cdots d^3\boldsymbol{r}_N \, P_t^{(N)}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \rho(\boldsymbol{r}) = P_t^{(1)}(\boldsymbol{r}), \tag{3.26}$$

i.e., the single-particle probability distribution. When two *independent* Bose gases are overlapping, no interference fringes are expected in this single-particle distribution $P_t^{(1)}(\boldsymbol{r})$, which represents the image obtained by accumulating and superposing many snapshots. An interference pattern, however, would be found in each snapshot, due to the higher-order correlations arising from the indistinguishability of identical atoms.

If fringes are present in a snapshot, we expect the density deviation

$$\delta\rho(\boldsymbol{r}) = \rho(\boldsymbol{r}) - \overline{\rho(\boldsymbol{r})} \tag{3.27}$$

to oscillate, which is captured by spikes in its Fourier transform

$$\delta\tilde{\rho}(\boldsymbol{k}) = \int d^3\boldsymbol{r}\,\delta\rho(\boldsymbol{r})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{3.28}$$

If a spike is found at $\boldsymbol{k}_f$, it reveals the oscillation of the density profile with fringe spacing $2\pi/k_f$.

Notice here that the phase (spatial offset) of the interference pattern varies randomly from snapshot to snapshot. This is actually unavoidable, in order to be consistent with the "independence" of the two gases: this random shift smears out the fringes in the average profile $\overline{\rho(\boldsymbol{r})}$, i.e., in the single-particle distribution $P_t^{(1)}(\boldsymbol{r})$, and the "independence" is recovered. In order to discard this random phase, we look at the square modulus of the Fourier spectrum, $|\delta\tilde{\rho}(\boldsymbol{k})|^2$. If sinusoidal patterns with a definite fringe spacing (with their random spatial offsets discarded) are *typical* among all possible snapshot profiles and are found in *almost all* snapshots, the spikes in the spectrum $|\delta\tilde{\rho}(\boldsymbol{k})|^2$ would remain even in its average over all possible realizations of $\{\boldsymbol{r}_1,\dots,\boldsymbol{r}_N\}$,

$$S_t(\boldsymbol{k}) = \overline{|\delta\tilde{\rho}(\boldsymbol{k})|^2} = \overline{|\tilde{\rho}(\boldsymbol{k})|^2} - \left|\overline{\tilde{\rho}(\boldsymbol{k})}\right|^2. \tag{3.29}$$

The *typicality* is characterized by the variance, or more generally, by the covariance

$$C_t(\boldsymbol{k},\boldsymbol{k}') = \overline{|\delta\tilde{\rho}(\boldsymbol{k})|^2|\delta\tilde{\rho}(\boldsymbol{k}')|^2} - \overline{|\delta\tilde{\rho}(\boldsymbol{k})|^2}\cdot\overline{|\delta\tilde{\rho}(\boldsymbol{k}')|^2}. \tag{3.30}$$

If the average spectrum $S_t(\boldsymbol{k})$ exhibits a nontrivial spike at $\boldsymbol{k} = \boldsymbol{k}_f$ with a *vanishingly small* covariance $C_t(\boldsymbol{k}_f,\boldsymbol{k}_f) \sim 0$, the sinusoidal pattern corresponding to the spike is expected to be observed in every snapshot. This is the idea: *if no fluctuation, we can speak of snapshot through average.*

By noting $\tilde{\rho}(\boldsymbol{k}) = \sum_{i=1}^N e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}/N$, one realizes that these quantities are given in terms of few-particle distribution functions. Indeed,

$$\overline{|\tilde{\rho}(\boldsymbol{k})|^2} = \frac{N-1}{N}I_t^{(2)}(\boldsymbol{k}) + \frac{1}{N}, \tag{3.31}$$

$$\overline{|\tilde{\rho}(\boldsymbol{k})|^2|\tilde{\rho}(\boldsymbol{k}')|^2} = \frac{(N-1)!}{N^3(N-4)!}I_t^{(4)}(\boldsymbol{k},\boldsymbol{k}') + O\big(\tfrac{1}{N}\big), \tag{3.32}$$

where

$$I_t^{(2)}(\boldsymbol{k}) = \int d^3\boldsymbol{r}_1\,d^3\boldsymbol{r}_2\,P_t^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2)e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)}, \tag{3.33}$$

$$I_t^{(4)}(\boldsymbol{k},\boldsymbol{k}')=\int d^3\boldsymbol{r}_1\,d^3\boldsymbol{r}_2\,d^3\boldsymbol{r}_3\,d^3\boldsymbol{r}_4\,P_t^{(4)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,\boldsymbol{r}_4)e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)+i\boldsymbol{k}'\cdot(\boldsymbol{r}_3-\boldsymbol{r}_4)}. \tag{3.34}$$

Namely, the average fringe contrast $S_t(\boldsymbol{k})$ of the $N$ particles is essentially ruled by the two-particle distribution $P_t^{(2)}$, while its fluctuation $C_t(\boldsymbol{k},\boldsymbol{k}')$ by $P_t^{(4)}$. We do not need to compute the $N$-particle distribution function $P_t^{(N)}$ in practice to discuss the average fringe spectrum $S_t(\boldsymbol{k})$ and the fluctuation $C_t(\boldsymbol{k},\boldsymbol{k}')$.

In general, the fluctuation of the snapshot profiles $\rho(\boldsymbol{r})$ is fully characterized by the generating functional

$$Z_t[\Phi]=\overline{e^{i\int d^3\boldsymbol{r}\,\Phi(\boldsymbol{r})\rho(\boldsymbol{r})}}. \tag{3.35}$$

When $N\gg 1$, it is cast into [1]

$$Z_t[\Phi]\simeq\langle{:}e^{\frac{i}{N}\int d^3\boldsymbol{r}\,\Phi(\boldsymbol{r})\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})}{:}\rangle_t, \tag{3.36}$$

where $:\cdots:$ denotes normal ordering. These are our tools for discussing the measurement-induced interference. See also [9–11].

We are going to compute the statistics of the density profiles $Z_t[\Phi]$ in (3.36), in particular, the average fringe spectrum $S_t(\boldsymbol{k})$ in (3.29) and the covariance $C_t(\boldsymbol{k},\boldsymbol{k}')$ in (3.30), for the ideal gases of bosonic atoms released from two separate harmonic traps.

### 3.2 Canonical and Grand Canonical Ensembles in Harmonic Traps

It is possible to generalize the calculation shown in Sec. 2 to compute the characteristic functional $W_N[J,J^*]$ for the canonical ensemble of noninteracting bosonic atoms in a 3D harmonic trap, in the continuum limit $\hbar\omega/k_BT\ll 1$ keeping $(\hbar\omega/k_BT)^3N$ finite. Here we assume, for simplicity, that the 3D harmonic trap is isotropic and the strength of the harmonic trapping is characterized by the single frequency $\omega$. The Hamiltonian of the systems reads

$$\hat{H}=\sum_{\boldsymbol{n}}\varepsilon_{\boldsymbol{n}}\hat{a}_{\boldsymbol{n}}^\dagger\hat{a}_{\boldsymbol{n}},\qquad \varepsilon_{\boldsymbol{n}}=\hbar\omega(n_x+n_y+n_z), \tag{3.37}$$

where $\boldsymbol{n}=(n_x,n_y,n_z)$ $(n_x,n_y,n_z=0,1,2,\ldots)$ are the quantum numbers labeling the energy states of the harmonic potential, and $\hat{a}_{\boldsymbol{n}}$ and $\hat{a}_{\boldsymbol{n}}^\dagger$ are the associated annihilation and the creation operators, satisfying the canonical commutation relations $[\hat{a}_{\boldsymbol{n}},\hat{a}_{\boldsymbol{n}'}^\dagger]=\delta_{\boldsymbol{n}\boldsymbol{n}'}$, etc. Then, the characteristic functional for the canonical ensemble of the noninteracting bosonic atoms in the harmonic trap is obtained, in the continuum limit, as [1]

$$W_N[J,J^*]\simeq\begin{cases}J_0(2\sqrt{N_0}|J_0|)\exp\left(-\displaystyle\sum_{\boldsymbol{n}\neq 0}\frac{|J_{\boldsymbol{n}}|^2}{e^{\beta\varepsilon_{\boldsymbol{n}}}-1}\right) & (T\le T_c,\ \mu=0),\\ \exp\left(-\displaystyle\sum_{\boldsymbol{n}}\frac{|J_{\boldsymbol{n}}|^2}{e^{\beta(\varepsilon_{\boldsymbol{n}}-\mu)}-1}\right) & (T>T_c,\ \mu<0),\end{cases} \tag{3.38}$$

where

$$\hat{W}[J,J^*]=e^{i\sum_{\boldsymbol{n}}J_{\boldsymbol{n}}\hat{a}_{\boldsymbol{n}}^\dagger}e^{i\sum_{\boldsymbol{n}}J_{\boldsymbol{n}}^*\hat{a}_{\boldsymbol{n}}}, \tag{3.39}$$

which is equivalent to (2.5) but in a different representation, and the condensation fraction is given by [4, 12]

$$\lambda=\frac{N_0}{N}\simeq\begin{cases}1-\left(\dfrac{T}{T_c}\right)^3 & (T\le T_c,\ \mu=0),\\ 0 & (T>T_c,\ \mu<0),\end{cases} \tag{3.40}$$

with the critical temperature

$$T_c=\frac{\hbar\omega}{k_B}\left(\frac{N}{\zeta(3)}\right)^{1/3}. \tag{3.41}$$

On the other hand, the grand canonical ensemble yields a different characteristic functional [1]

$$W_G[J,J^*]\simeq\begin{cases}e^{-N_0|J_0|^2}\exp\left(-\displaystyle\sum_{\boldsymbol{n}\neq0}\frac{|J_{\boldsymbol{n}}|^2}{e^{\beta\varepsilon_{\boldsymbol{n}}}-1}\right) & (T\le T_c,\ \mu=0),\\ \exp\left(-\displaystyle\sum_{\boldsymbol{n}}\frac{|J_{\boldsymbol{n}}|^2}{e^{\beta(\varepsilon_{\boldsymbol{n}}-\mu)}-1}\right) & (T>T_c,\ \mu<0).\end{cases} \tag{3.42}$$

### 3.3 Statistics of the Density Profiles of the Overlapping Gases

We are now ready to compute the statistics of the density profiles, $Z_t[\Phi]$ in (3.36), of the overlapping gases released from two identical harmonic traps "$L$" and "$R$" separated by a vector $\boldsymbol{d}$. We assume that each gas contains exactly $N$ atoms and is described as a canonical ensemble at temperature $T$. Before the release from the traps, there is no correlation between the two gases, in particular no phase correlation, and no exchange of atoms between the two traps: the two gases are "independent." The latter condition is mathematically represented by the commutativity between the bosonic operators $\hat{a}_{\boldsymbol{n}}^{(L)}$, $\hat{a}_{\boldsymbol{n}}^{(L)\dagger}$ for the $L$ trap and $\hat{a}_{\boldsymbol{n}}^{(R)}$, $\hat{a}_{\boldsymbol{n}}^{(R)\dagger}$ for the $R$ trap, which is equivalent to the condition that the eigenfunctions $\varphi_{\boldsymbol{n}}^{(L)}(\boldsymbol{r})$ of the $L$ trap are all orthogonal to the eigenfunctions $\varphi_{\boldsymbol{n}}^{(R)}(\boldsymbol{r})$ of the $R$ trap. This is at least approximately valid, if the two traps are well separated and the spatial overlaps between the eigenfunctions $\varphi_{\boldsymbol{n}}^{(L)}(\boldsymbol{r})$ and $\varphi_{\boldsymbol{n}}^{(R)}(\boldsymbol{r})$ of the relevant low-energy levels ($\varepsilon_{\boldsymbol{n}}\lesssim k_BT$) for the finite-temperature gases are negligible. Then, the field operator $\hat{\psi}(\boldsymbol{r})$ is expanded by those eigenfunctions (at least up to the relevant levels $\varepsilon_{\boldsymbol{n}}\lesssim k_BT$) as

$$\hat{\psi}(\boldsymbol{r})=\sum_{\ell=L,R}\sum_{\boldsymbol{n}}\hat{a}_{\boldsymbol{n}}^{(\ell)}\varphi_{\boldsymbol{n}}^{(\ell)}(\boldsymbol{r}). \tag{3.43}$$

The state of the pair of gases before the release from the traps is the product state of the canonical states for the $L$ and $R$ gases, and accordingly, the characteristic functional for the pair is given by the product of (3.38) for $L$ and $R$:

$$\begin{aligned}W_{N+N}[J,J^*]&=\langle e^{i\sum_\ell\sum_{\boldsymbol{n}}J_{\boldsymbol{n}}^{(\ell)}\hat{a}_{\boldsymbol{n}}^{(\ell)\dagger}}e^{i\sum_\ell\sum_{\boldsymbol{n}}J_{\boldsymbol{n}}^{(\ell)*}\hat{a}_{\boldsymbol{n}}^{(\ell)}}\rangle_{N+N}\\&=\langle e^{i\sum_{\boldsymbol{n}}J_{\boldsymbol{n}}^{(L)}\hat{a}_{\boldsymbol{n}}^{(L)\dagger}}e^{i\sum_{\boldsymbol{n}}J_{\boldsymbol{n}}^{(L)*}\hat{a}_{\boldsymbol{n}}^{(L)}}\rangle_N\langle e^{i\sum_{\boldsymbol{n}}J_{\boldsymbol{n}}^{(R)}\hat{a}_{\boldsymbol{n}}^{(R)\dagger}}e^{i\sum_{\boldsymbol{n}}J_{\boldsymbol{n}}^{(R)*}\hat{a}_{\boldsymbol{n}}^{(R)}}\rangle_N.\end{aligned} \tag{3.44}$$

In the coordinate representation, it reads

$$\simeq\begin{cases}\displaystyle\int_{-\pi}^{\pi}\frac{d\theta_L}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_R}{2\pi}e^{2i\sqrt{N}\,\mathrm{Re}\sum_\ell\int d^3\boldsymbol{r}\,\alpha^{(\ell)}(\boldsymbol{r})e^{i\theta_\ell}J^*(\boldsymbol{r})}\\ \displaystyle\qquad\times\exp\!\left(-2N\int d^3\boldsymbol{r}\,d^3\boldsymbol{r}'\,J^*(\boldsymbol{r})\mathcal{F}'(\boldsymbol{r},\boldsymbol{r}')J(\boldsymbol{r}')\right)\\ \qquad\qquad\qquad(T\le T_c,\ \mu=0),\\ \displaystyle\exp\!\left(-2N\int d^3\boldsymbol{r}\,d^3\boldsymbol{r}'\,J^*(\boldsymbol{r})\mathcal{F}(\boldsymbol{r},\boldsymbol{r}')J(\boldsymbol{r}')\right)\qquad(T>T_c,\ \mu<0),\end{cases}\tag{3.45}$$

where

$$J_{\boldsymbol{n}}^{(\ell)}=\int d^3\boldsymbol{r}\,\varphi_{\boldsymbol{n}}^{(\ell)*}(\boldsymbol{r})J(\boldsymbol{r}),\qquad J(\boldsymbol{r})=\sum_{\ell=L,R}J_{\boldsymbol{n}}^{(\ell)}\varphi_{\boldsymbol{n}}^{(\ell)}(\boldsymbol{r}),\tag{3.46}$$

and

$$\alpha^{(\ell)}(\boldsymbol{r})=\sqrt{\lambda}\varphi_0^{(\ell)}(\boldsymbol{r}),\qquad\mathcal{F}'(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{2N}\sum_{\ell=L,R}\sum_{\boldsymbol{n}\neq0}\varphi_{\boldsymbol{n}}^{(\ell)}(\boldsymbol{r})\frac{1}{e^{\beta\varepsilon_{\boldsymbol{n}}}-1}\varphi_{\boldsymbol{n}}^{(\ell)*}(\boldsymbol{r}'),\tag{3.47}$$

$$\mathcal{F}(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{2N}\sum_{\ell=L,R}\sum_{\boldsymbol{n}}\varphi_{\boldsymbol{n}}^{(\ell)}(\boldsymbol{r})\frac{1}{e^{\beta(\varepsilon_{\boldsymbol{n}}-\mu)}-1}\varphi_{\boldsymbol{n}}^{(\ell)*}(\boldsymbol{r}').\tag{3.48}$$

$\alpha^{(\ell)}(\boldsymbol{r})$ is the wave function of condensate $\ell=L,R$, while $\mathcal{F}'(\boldsymbol{r},\boldsymbol{r}')$ and $\mathcal{F}(\boldsymbol{r},\boldsymbol{r}')$ are the single-particle density matrices of the noncondensed components. We shall use the same symbol $\mathcal{F}$ for the density matrix below the critical temperature,

$$\mathcal{F}(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{2}\sum_{\ell=L,R}\alpha^{(\ell)}(\boldsymbol{r})\alpha^{(\ell)*}(\boldsymbol{r}')+\mathcal{F}'(\boldsymbol{r},\boldsymbol{r}')\qquad(T\le T_c,\ \mu=0).\tag{3.49}$$

Starting from this initial condition, the gases are released from the traps and expand freely in the absence of collision among the atoms. In the Heisenberg picture, the field operator (3.43) simply evolves as

$$\hat{\psi}(\boldsymbol{r},t)=\sum_{\ell=L,R}\sum_{\boldsymbol{n}}\hat{a}_{\boldsymbol{n}}^{(\ell)}\varphi_{\boldsymbol{n}}^{(\ell)}(\boldsymbol{r},t).\tag{3.50}$$

Therefore, in order to implement the time evolution, we have only to replace $\varphi_{\boldsymbol{n}}^{(\ell)}(\boldsymbol{r})\to\varphi_{\boldsymbol{n}}^{(\ell)}(\boldsymbol{r},t)=e^{(i\hbar t/2m)\nabla^2}\varphi_{\boldsymbol{n}}^{(\ell)}(\boldsymbol{r})$, and accordingly, $\alpha^{(\ell)}(\boldsymbol{r})\to\alpha_t^{(\ell)}(\boldsymbol{r})$, $\mathcal{F}'(\boldsymbol{r},\boldsymbol{r}')\to\mathcal{F}_t'(\boldsymbol{r},\boldsymbol{r}')$ and $\mathcal{F}(\boldsymbol{r},\boldsymbol{r}')\to\mathcal{F}_t(\boldsymbol{r},\boldsymbol{r}')$, in (3.45). Then, the generating functional of the density profiles, $Z_t[\Phi]$ in (3.36), of such an expanding cloud is generated as [1]

$$Z_{N+N,t}[\Phi]=e^{\frac{i}{2N}\int d^3\boldsymbol{r}\,\frac{\delta}{\delta iJ(\boldsymbol{r})}\Phi(\boldsymbol{r})\frac{\delta}{\delta iJ^*(\boldsymbol{r})}}W_{N+N,t}[J,J^*]\Big|_{J,J^*=0}$$

$$\simeq\begin{cases}\displaystyle\frac{1}{\mathrm{Det}[1-i\Phi(\hat{\boldsymbol{r}})\hat{\mathcal{F}}_t']}e^{\frac{i}{2}\sum_\ell\langle\alpha_t^{(\ell)}|\frac{1}{\Phi^{-1}(\hat{\boldsymbol{r}})-i\hat{\mathcal{F}}_t'}|\alpha_t^{(\ell)}\rangle}\\ \displaystyle\qquad\times\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{\frac{i}{2}\left(\langle\alpha_t^{(L)}|\frac{1}{\Phi^{-1}(\hat{\boldsymbol{r}})-i\hat{\mathcal{F}}_t'}|\alpha_t^{(R)}\rangle e^{i\theta}+\langle\alpha_t^{(R)}|\frac{1}{\Phi^{-1}(\hat{\boldsymbol{r}})-i\hat{\mathcal{F}}_t'}|\alpha_t^{(L)}\rangle e^{-i\theta}\right)}\\ \qquad\qquad\qquad(T\le T_c,\ \mu=0),\\ \displaystyle\frac{1}{\mathrm{Det}[1-i\Phi(\hat{\boldsymbol{r}})\hat{\mathcal{F}}_t]}\qquad(T>T_c,\ \mu<0),\end{cases}\tag{3.51}$$

where an abstract notation has been introduced by

$$\alpha_t^{(\ell)}(\boldsymbol{r}) = \langle \boldsymbol{r}|\alpha_t^{(\ell)}\rangle, \qquad \mathcal{F}_t'(\boldsymbol{r},\boldsymbol{r}') = \langle \boldsymbol{r}|\hat{\mathcal{F}}_t'|\boldsymbol{r}'\rangle, \qquad \mathcal{F}_t(\boldsymbol{r},\boldsymbol{r}') = \langle \boldsymbol{r}|\hat{\mathcal{F}}_t|\boldsymbol{r}'\rangle, \tag{3.52}$$

and

$$\hat{r}|\boldsymbol{r}\rangle = \boldsymbol{r}|\boldsymbol{r}\rangle, \qquad \langle \boldsymbol{r}|\boldsymbol{r}'\rangle = \delta^3(\boldsymbol{r}-\boldsymbol{r}'), \qquad \int d^3\boldsymbol{r}\,|\boldsymbol{r}\rangle\langle\boldsymbol{r}| = 1. \tag{3.53}$$

This generating functional should be compared with that for the grand canonical ensembles [1],

$$Z_{G+G,t}[\Phi] \simeq \frac{1}{\mathrm{Det}[1-i\Phi(\hat{\boldsymbol{r}})\hat{\mathcal{F}}_t]}. \tag{3.54}$$

Below the critical temperature $T \le T_c$, the density profiles fluctuate in different ways for the canonical and the grand canonical ensembles.

## 3.4 Snapshot Interference and Condensation

We are finally in a position to discuss the snapshot interference of the overlapping gases of noninteracting bosonic atoms released from two spatially separated harmonic traps.

Observe first the average profile

$$\overline{\rho(\boldsymbol{r})} = \left.\frac{\delta Z_{N+N,t}[\Phi]}{\delta i\Phi(\boldsymbol{r})}\right|_{\Phi=0} = \mathcal{F}_t(\boldsymbol{r},\boldsymbol{r}), \tag{3.55}$$

and recall the definition of $\mathcal{F}_t(\boldsymbol{r},\boldsymbol{r}')$ in (3.47)–(3.49). It is just the sum of the density operators of the two gases $L$ and $R$, and no interference is observed in this quantity. However, interference fringes are found in each snapshot.

### 3.4.1 At zero temperature $T = 0$

Let us look at the zero temperature case $T = 0$. In this case, the average profile (3.55) is reduced to

$$\overline{\rho(\boldsymbol{r})} = \frac{1}{2}\left(|\alpha_t^{(L)}(\boldsymbol{r})|^2 + |\alpha_t^{(R)}(\boldsymbol{r})|^2\right). \tag{3.56}$$

On the other hand, the average fringe spectrum (3.29) reads

$$S_t(\boldsymbol{k}) = \frac{1}{4}\left(|\chi_t(\boldsymbol{k})|^2 + |\chi_t(-\boldsymbol{k})|^2\right), \tag{3.57}$$

while the covariance (3.30)

$$C_t(\boldsymbol{k},\boldsymbol{k}') = \frac{1}{8}\,\mathrm{Re}[\chi_t^*(\boldsymbol{k})\chi_t^*(-\boldsymbol{k})\chi_t(-\boldsymbol{k}')\chi_t(\boldsymbol{k}')], \tag{3.58}$$

where

$$\chi_t(\boldsymbol{k}) = \langle \alpha_t^{(R)}|e^{-i\boldsymbol{k}\cdot\hat{\boldsymbol{r}}}|\alpha_t^{(L)}\rangle, \tag{3.59}$$

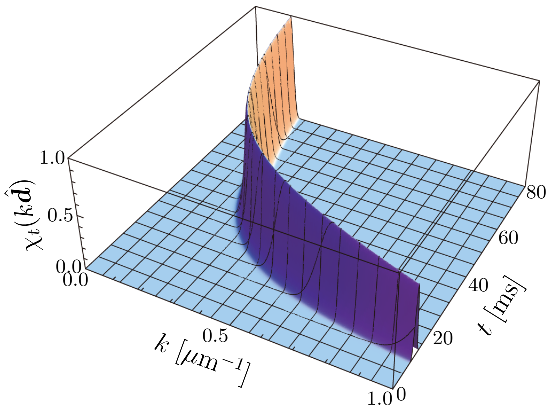


Figure 1: The time evolution of the interference spectrum $\chi_t(\boldsymbol{k})$ in (3.61) between two pure condensates at zero temperature $T = 0$. The two condensates, each containing $N = 5 \times 10^6$ Na atoms, are released from two harmonic traps of trapping frequency $\omega = 1.6\,\mathrm{kHz}$, separated by a distance $d = 30\,\mu\mathrm{m}$. The condensates expand, overlap, and exhibit an interference pattern in each snapshot. The critical temperature of the gas trapped in this harmonic potential is estimated by (3.41) to be $T_c = 2.0\,\mu\mathrm{K}$.

which is the Fourier transform of the "interference term" $\alpha_t^{(R)*}(\boldsymbol{r})\alpha_t^{(L)}(\boldsymbol{r})$ between the two condensate wave functions $\alpha_t^{(L)}(\boldsymbol{r})$ and $\alpha_t^{(R)}(\boldsymbol{r})$. For the harmonic traps, they are given by

$$\alpha_t^{(L/R)}(\boldsymbol{r}) = \left(\frac{m\omega}{\pi\hbar(1+i\omega t)^2}\right)^{3/4} e^{-m\omega(\boldsymbol{r}\pm\boldsymbol{d}/2)^2/2\hbar(1+i\omega t)}, \tag{3.60}$$

with $\boldsymbol{d}$ representing the displacement between the two traps, and

$$\chi_t(\boldsymbol{k}) = e^{-\hbar[\boldsymbol{k}^2+(\boldsymbol{k}\omega t - m\omega\boldsymbol{d}/\hbar)^2]/4m\omega}. \tag{3.61}$$

The time evolution of $\chi_t(\boldsymbol{k})$ is shown in Fig. 1. Two sharp peaks grow in the average spectrum $S_t(\boldsymbol{k})$ in (3.57) at

$$\boldsymbol{k} = \pm\boldsymbol{k}_f, \qquad \boldsymbol{k}_f = \frac{m\boldsymbol{d}}{\hbar t}. \tag{3.62}$$

The peaks become sharper and higher as the time of flight increases. The covariance $C_t(\boldsymbol{k},\boldsymbol{k}')$ in (3.58), on the other hand, is vanishingly small for any $(\boldsymbol{k},\boldsymbol{k}')$, since the peaks of $\chi_t(\boldsymbol{k})$ and $\chi_t(-\boldsymbol{k})$ are well separated. This means that there is no fluctuation in the fringe spectrum and an interference pattern with the fringe spacing $2\pi/k_f$ is *certainly observed in every snapshot*. Note that the visibility of the interference pattern is essentially ruled by the height of the spectrum $S_t(\pm\boldsymbol{k}_f)$, and its asymptotic height $S_t(\boldsymbol{k}_f) \to 1/4$ corresponds to the perfect visibility.[2]

Note also that the characteristic functional for the canonical ensemble, $W_N[J, J^*]$ in (3.38), gives

$$\langle\hat{a}_0\rangle_N = 0. \tag{3.63}$$

[2]The Fourier transform of $\cos\boldsymbol{k}_f\cdot\boldsymbol{r} = \frac{1}{2}(e^{i\boldsymbol{k}_f\cdot\boldsymbol{r}} + e^{-i\boldsymbol{k}_f\cdot\boldsymbol{r}})$ exhibits peaks at $\boldsymbol{k} = \pm\boldsymbol{k}_f$ of height $1/2$, whose square gives $1/4$.

That is, the U(1) symmetry of the system is not broken. Actually, no interference is observed in the single-particle distribution (3.56), but an interference pattern appears in each snapshot.

### 3.4.2 At finite temperature $T > 0$

At a finite temperature $T > 0$, formulas for the average fringe spectrum $S_t(\boldsymbol{k})$ and the covariance $C_t(\boldsymbol{k}, \boldsymbol{k}')$ are available for a long time of flight $t$ [1]. They exhibit sharp peaks at $\boldsymbol{k} = 0$ and $\pm\boldsymbol{k}_f$, whose heights are given by

$$\begin{cases} \overline{\tilde{\rho}(0)} \sim 1, \\ \overline{\tilde{\rho}(\boldsymbol{k}_f)} \sim 0, \end{cases} \qquad \begin{cases} S_t(\boldsymbol{k}_f) \sim \dfrac{1}{4}\operatorname{Tr}\{\hat{\mathcal{F}}_0^2\}, \\ S_t(0) \sim \dfrac{1}{2}\operatorname{Tr}\{\hat{\mathcal{F}}_0'^2\}, \end{cases} \qquad \begin{cases} C_t(\boldsymbol{k}_f, \boldsymbol{k}_f) \sim \dfrac{1}{16}(2\operatorname{Tr}\{\hat{\mathcal{F}}_0^4\} + \operatorname{Tr}\{\hat{\mathcal{F}}_0^2\}^2 - 3\lambda^4), \\ C_t(0,0) \sim \dfrac{1}{4}(3\operatorname{Tr}\{\hat{\mathcal{F}}_0'^4\} + 2\operatorname{Tr}\{\hat{\mathcal{F}}_0'^2\}^2), \\ C_t(\boldsymbol{k}_f, 0) \sim \dfrac{3}{8}\operatorname{Tr}\{\hat{\mathcal{F}}_0'^4\}, \end{cases} \tag{3.64}$$

where $\hat{\mathcal{F}}_0$ is the single-particle density operator of the gas in each trap, and $\hat{\mathcal{F}}_0' = \hat{\mathcal{F}}_0 - |\alpha_0\rangle\langle\alpha_0|$ is its thermal part with $|\alpha_0\rangle$ being its condensed component. These expressions are valid over the whole range of temperature $T$, across the critical temperature $T_c$.

Recall here that $\operatorname{Tr}\{\hat{\mathcal{F}}_0^2\}$ is the "purity" of each gas, and the average fringe spectrum $S_t(\boldsymbol{k}_f)$ in (3.64) is given by this purity. The purity is vanishingly small $\operatorname{Tr}\{\hat{\mathcal{F}}_0^2\} \sim 0$ in the absence of condensate above the critical temperature $T > T_c$, while it becomes $\operatorname{Tr}\{\hat{\mathcal{F}}_0^2\} \sim O(1)$ as the ground state is occupied by a macroscopic number of atoms below the critical temperature $T \le T_c$, approaching $\operatorname{Tr}\{\hat{\mathcal{F}}_0^2\} = 1$ for pure condensation at $T = 0$. The purity is a good measure of condensation and is adopted for a criterion of the Bose-Einstein condensation by Penrose and Onsager [13]. The formula for the average fringe spectrum $S_t(\boldsymbol{k}_f)$ in (3.64) explicitly clarifies *the connection between the condensation and the interference*, and *the importance of the condensation for the interference*. The purity

$$\operatorname{Tr}\{\hat{\mathcal{F}}_0^2\} = \lambda^2 + \operatorname{Tr}\{\hat{\mathcal{F}}_0'^2\} \tag{3.65}$$

is different from $\lambda^2$ only by $\operatorname{Tr}\{\hat{\mathcal{F}}_0'^2\} \simeq O(1/N)$ [1], and therefore, the purity is essentially given by the square of the condensation fraction $\lambda^2$ [11]. See Fig. 2, where the average fringe spectrum $S_t(\boldsymbol{k}_f)$ is plotted as a function of the temperature $T$.

The fluctuation of the fringe spectrum (relative to the average), on the other hand, is estimated to be

$$\frac{C_t(\boldsymbol{k}_f, \boldsymbol{k}_f)}{S_t^2(\boldsymbol{k}_f)} \sim 1 - \frac{\lambda^4 - 2\operatorname{Tr}\{\hat{\mathcal{F}}_0'^4\}}{(\lambda^2 + \operatorname{Tr}\{\hat{\mathcal{F}}_0'^2\})^2} \simeq \begin{cases} O(1/N) & (T \le T_c), \\ 1 + O(1/N) & (T > T_c), \end{cases} \tag{3.66}$$

by noting that

$$\lambda \simeq \begin{cases} O(1), \\ O(1/N), \end{cases} \qquad \operatorname{Tr}\{\hat{\mathcal{F}}_0'^4\} \simeq \begin{cases} O(1/N^2) & (T \le T_c), \\ O(1/N^3) & (T > T_c), \end{cases} \tag{3.67}$$

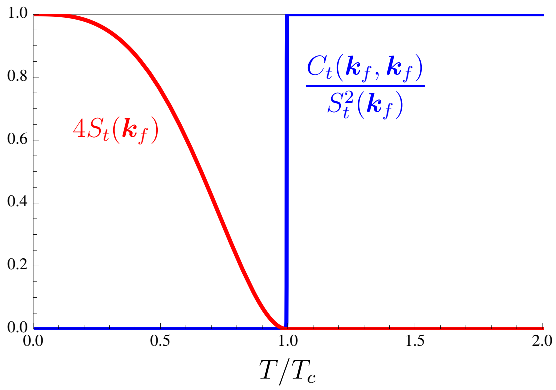


Figure 2: The average and the fluctuation of the snapshot interference spectrum, $S_t(\boldsymbol{k}_f)$ and $C_t(\boldsymbol{k}_f,\boldsymbol{k}_f)$ given in (3.64), as functions of the temperature of the gases $T$. The parameters are the same as in Fig. 1. The relevant quantities $\lambda$, $\mathrm{Tr}\{\hat{\mathcal{F}}_0'^2\}$, and $\mathrm{Tr}\{\hat{\mathcal{F}}_0'^4\}$ are numerically evaluated without resort to the continuum limit.

and $\mathrm{Tr}\{\hat{\mathcal{F}}_0'^2\} \simeq O(1/N)$ for the whole temperature range [1]. The fluctuation is vanishingly small below the critical temperature $T \le T_c$ [10], while it is nonvanishing above $T > T_c$. As shown in Fig. 2, the fluctuation abruptly changes at the critical temperature $T_c$. In particular, the interference spectrum does not fluctuate at any temperature below the critical temperature $T \le T_c$, and in this range, *the interference pattern with fringe contrast $\lambda$ is certainly observed in every snapshot.*

If the gases are described as grand canonical ensembles, instead of the canonical ensembles, the statistics of the snapshot profiles are given by $Z_{G+G,t}[\Phi]$ in (3.54), and we end up with different conclusion from the above. While the average fringe spectrum $S_t(\boldsymbol{k}_f)$ remains unchanged, the variance $C_t(\boldsymbol{k}_f,\boldsymbol{k}_f)$ exhibits different fluctuation for the grand canonical ensembles:

$$\frac{C_t(\boldsymbol{k}_f,\boldsymbol{k}_f)}{S_t^2(\boldsymbol{k}_f)} \sim 1 + \frac{2\,\mathrm{Tr}\{\hat{\mathcal{F}}_0^4\}}{\mathrm{Tr}\{\hat{\mathcal{F}}_0^2\}^2} \simeq \begin{cases} 3 + O(1/N) & (T \le T_c), \\ 1 + O(1/N) & (T > T_c). \end{cases} \tag{3.68}$$

The fringe spectrum largely fluctuates below the critical temperature $T \le T_c$, in contrast to the vanishing fluctuation with the canonical ensembles in (3.66) and in Fig. 2. If the gases are described as the grand canonical ensembles, we are not sure whether an interference pattern is observed in a snapshot photo.

The main difference between the canonical and the grand canonical ensembles is the fluctuation of the total number of atoms. In the case of the canonical ensemble, it does not fluctuate, $(\Delta N)_N^2 = 0$, since the total number of atoms is fixed at $N$. In the case of the grand canonical ensemble, on the other hand, it is estimated to be

$$\frac{(\Delta N)_G^2}{N^2} \simeq \begin{cases} \lambda^2 + O(1/N) & (T \le T_c), \\ O(1/N) & (T > T_c), \end{cases} \tag{3.69}$$

and the total number of atoms becomes fluctuating below the critical temperature $T \leq T_c$. Although usually the canonical and the grand canonical ensembles coincide in the thermodynamical limit $N \to \infty$, it is not the case in the presence of condensate. This difference leads to the difference in the fluctuation of the fringe spectrum in (3.66) and (3.68).

On the other hand, the fluctuations of the energies of the canonical and the grand canonical ensembles are both negligibly small for large $N$,

$$\frac{(\Delta E)^2}{E^2} \sim O(1/N), \tag{3.70}$$

irrespective of the presence of condensate, since the energy of the condensate is zero and does not contribute to the energy $E$ and its fluctuation $\Delta E$. Therefore, in the large $N$ limit, the canonical ensemble well mimics the *microcanonical ensemble*, in which the total number of atoms $N$ and the total energy $E$ are fixed, and the results of the present paper for the canonical ensemble are all (approximately) valid for the interference of two independent microcanonical gases.

# 4 Summary

We have discussed the interference of two independently prepared Bose-Einstein condensates. When the number of atoms in each gas is fixed and the phases of the gases are uncertain, no interference is expected in the single-particle distribution. However, the interference is observed in each single snapshot. We have presented tools for studying such snapshot interference, *the measurement-induced interference*. The idea is to discard the randomness in the snapshot profiles (the spatial offsets of the interference patterns) and to focus on the *typical* feature in the interference patterns (the fringe spacing). If the sinusoidal patterns with a definite fringe spacing is actually typical, the fringe spectrum with the random offset discarded does not fluctuate, and we are allowed to speak of the snapshot interference through its average.

We have described the two gases as *canonical ensembles* with the numbers of atoms fixed at $N$ individually. We have shown that the covariance of the fringe spectrum over all possible snapshot profiles is vanishingly small below the critical temperature of the Bose-Einstein condensation: the interference pattern whose fringe contrast is characterized by the average fringe spectrum is *certainly observed in every snapshot in the presence of condensates.* The fringe contrast becomes stronger as the temperature is lowered and the condensation fraction is increased. This clarifies the importance of the Bose-Einstein condensation to the interference of independent Bose-Einstein condensates. The knowledge I acquired from Prof. Tasaki's lecture on how to characterize the canonical ensemble of bosonic atoms has enabled us to carry out this analysis.

Like this example, the coherence phenomena of quantum many-body systems in general would be explained on the basis of the idea of the measurement-indued coherence, without resort to the idea of the spontaneous symmetry breaking. This is actually true also for fermionic systems: the relative phase between two independent superconductors is built up by measurement, even though their U(1) symmetries are not broken and their phases are uncertain beforehand. It would be interesting to explore the possibility of explaining various phenomena, in which the symmetry breaking plays an essential role, on the basis of the measurement-induced coherence.

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